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On Certain Ruled Surfaces of the Fourth Order.

By Thomas F. Holgate.

Introduction.

The Ruled Surface of the Fourth Order has been studied to a considerable extent both by analytical and by synthetic processes. Perhaps the most important and complete analytical treatment of the surface is by Cayley in his second and third "Memoirs on Skew Surfaces, otherwise Scrolls" (Philosophical Transactions, 1864 and 1869, respectively). In these the surface is considered from a purely algebraic point of view, and the author is concerned chiefly with the determination of the possible different varieties of the surface. These are distinguished by the nature of the double curve, and eight species are enumerated in the second memoir, to which two others are added in the third memoir.

The principal synthetic treatment of the surface is by Cremona (Sulle superficie gobbe di quarto grado, Mem. della R. Istoria di Bologna, Serie II, T. VIII), who makes his first distinction between those surfaces which are of deficiency zero and those of deficiency unity. Of the former he finds ten species, of the latter, two, thus obtaining two surfaces not observed by Cayley. This enumeration of twelve species is, I believe, generally conceded to be complete. The surface has also been studied to a greater or less extent by Rohn, Salmon, Chasles, Reye and others.

In the following paper I consider those species of the surface which may be generated by two projective sheaves of planes of the second order. These all admit a trinodal quartic section, and are consequently of deficiency zero. A direct demonstration was necessary at the outset that two such sheaves of planes generate in general a Ruled Surface of the Fourth Order, and after this follows a discussion of the different species in what appears to be the most natural order of succession. The reciprocal of each of these surfaces and the bitangent torse accompanying each are pointed out, and an enumeration of the subforms of the different species is made which appears to be complete. In order to avoid frequent references it was deemed advisable to preface the treatment of the Ruled Surface with a brief statement of those properties of the line congruences of which constant use is made in the subsequent work.

§1.—*The Line Congruence of the First Order.*

1. Let there be given two collinear bundles of rays and planes, designated by their centres S and S', which are neither concentric nor in perspective position, and which, in general, have no self-corresponding ray or plane. Corresponding forms in these bundles—for example, sheaves of lines which lie in corresponding planes, sheaves of planes whose axes are corresponding rays, corresponding cones or sheaves of planes of the second order—are projectively related to one another.* The line system, Σ, generated by the intersection of corresponding planes in the bundles S and S', is a congruence of the first order, since through every point in space there passes one, and in general but one, line of this congruence.† If in any point P two lines of the congruence intersect, then is P a "singular point" with respect to the congruence, or more briefly, a singular point of the congruence. For in this point must also a pair of corresponding rays of the bundles intersect, and consequently an infinite number of lines of the congruence which lie on a cone of the second order, and which together pass through all possible singular points of the congruence.‡

If α and α' be any two corresponding planes in the bundles S and S', the sheaves of lines of the first order lying in these planes, whose centres are the centres of the bundles, are projective to one another, and hence the line of intersection of these planes contains two projective ranges of points which lie in one another. These have two self-corresponding points which may be either real and distinct, coincident or imaginary.|| These self-corresponding points are singular points of the congruence, since in them corresponding rays of the bundles intersect. Every line of the system Σ, therefore, contains two singular points, and conversely, every straight line which passes through two singular points is a line of the congruence, since it is the intersection of a pair of corresponding planes in the bundles. No straight line contains more than two singular points,

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* Reye, Geometrie der Lage, Ab. II, S. 5.

† Reye, G. d. L., II, S. 84.

‡ Reye, G. d. L., II, S. 85.

|| Reye, G. d. L., I, S. 169.

for then would every point of this line be a singular point, and the bundles would contain at least one self-corresponding plane, namely, the planes determined by the ray SS' and its corresponding ray in each of the bundles would coincide, and would consequently be self-corresponding, which is contrary to supposition.

It is readily seen that singular points of the congruence always exist, since the ray SS' of the bundle S must meet its corresponding ray of S' in the point S', and hence S' is a singular point. Similarly, S is a singular point of the congruence. The lines of the system Σ which pass through S lie, therefore, on a cone of the second order, K, which passes through all singular points of the congruence, in particular through the point S'. Similarly, the lines of the system which pass through S' lie on a cone of the second order, K', which passes through all singular points of the congruence, in particular through the point S. Hence all the singular points of the congruence lie on the curve of intersection of the cones K and K'. This curve, which we shall denote by k^3, passes through both the points S and S', and is of the third order, since an arbitrary plane intercepts it in at least one, but in general and at most in three points.* Every point of k^3 is a singular point of the congruence, since in each point of this curve two rays of the cones, i. e. two lines of the congruence intersect. The points S and S' are wholly arbitrary points of k^3,† and therefore out of any two points of this curve the lines of the congruence are projected by two collinear bundles, and the curve itself, by two projective cones of the second order whose rays are lines of the congruence.

If γ be any plane through S and γ' its corresponding plane through S', these will intersect in a line of the congruence which will contain two singular points, i. e. will intersect k^3 in two points, which, however, are real and distinct, coincident or imaginary, according as γ and γ' contain two real and distinct, two coincident or two imaginary rays of the cones K and K' respectively. Every line of the congruence Σ may therefore be considered a chord of the curve k^3. Those chords which meet k^3 in two real and distinct points we shall denote as "actual chords"; those which meet k^3 in two consecutive points, as "tangents," and those which meet the curve in two imaginary points, as "ideal chords" of this curve. No straight line can meet k^3 in more than two points, and a line which meets k^3 in a single point does not belong to this congruence.

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* Reye, G. d. L., II, S. 88.

† Reye, G. d. L., II, S. 90.

Since those planes which contain two consecutive rays of a cone of the second order form a sheaf of planes of the second order,[*] it follows immediately that the tangent system of the twisted cubic curve k^3 is projected out of any two points of the curve, S and S', by means of two projective sheaves of planes of the second order.

It has been already observed that an arbitrary plane cuts k^3, in general, and at most in three points, and it is evident that if these three points are all real and distinct, the plane will contain three actual chords of the curve. If two of these points approach indefinitely near to each other and finally coincide, i. e. if the plane meet k^3 in two points at one of which it touches the curve, two of the three chords lying in the plane will coincide and the third will become a tangent at the point of contact of the plane. If all three points of intersection coincide, i. e. if the plane osculate the curve at a certain point, then the three chords will fall together and will coincide with the tangent at the point of osculation. Finally, two of the points of intersection may become imaginary, in which case the plane will contain but one real chord, and that an ideal chord.

Thus the bundles S and S' generate in the most general case a line congruence of the first order and third class, whose singular points lie on a non-degenerate twisted cubic curve passing through the centres of the bundles.

2. Suppose now that the collinear bundles S and S' have one self-corresponding plane, α, but no self-corresponding ray. The projective sheaves of lines lying in α generate a curve of the second order, k^2, passing through S and S', each of whose points is a singular point of the congruence arising from these bundles. The remaining singular points lie in a straight line, k, which meets the singular conic, but which does not lie in a plane with it.[†] This singular line, k, is itself a line of the congruence, being the intersection of a pair of corresponding planes of S and S'.

Each line of the congruence meets both k^2 and k, since every plane of the bundles S and S' must intercept the conic in some second point, and must also intercept the straight line. Conversely, every straight line which meets both k^2 and k is a line of the congruence. Those lines of the congruence which pass through any point of k^2 lie in a plane with k; in particular, the lines of the congruence which pass through O, the point of intersection of k^2 and k, lie in that plane through k which is tangent to k^2. The lines of the congruence which pass through any point of the singular line k lie on a cone of the second order of which k is always one ray, and which projects the conic k^2; only for the point O does this cone break up into two planes, namely, the self-corresponding plane α and the plane through k tangent to k^2.

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[*] Reye, G. d. L., I, S. 87.

[†] Reye, G. d. L., II, S. 85.

Out of any point of k the congruence is projected by a sheaf of planes of the first order, each plane of the sheaf containing an infinite number of lines of the system, while out of any two points of the conic k^2 the congruence is projected by means of two collinear bundles,* so that S and S' may be any points whatever of k^2, the point O being the only exceptional point.

Through every point in space there passes one, and in general but one, line of this congruence, while an arbitrary plane contains in general two of these lines, namely, the lines joining the point of intersection of the plane with k to its two points of intersection with the conic k^2. Thus the congruence generated by the collinear bundles S and S' which have a self-corresponding plane, but no other self-corresponding element, is of the first order and second class, its singular points lying on a conic and a straight line which meet but which do not lie in the same plane.

3. If the two bundles S and S' be so situated as to have the ray joining their centres a self-corresponding ray, then must every plane through this ray correspond again to a plane through this ray. In general, there will be two self-corresponding planes, α_1 and α_2, passing through the ray SS', which, however, may be either real and distinct, coincident or imaginary.

Consider, first, the self-corresponding planes, α_1 and α_2, to be real and distinct. The pairs of projective sheaves of lines lying in these planes generate straight lines, k_1 and k_2, respectively, every point of which is a singular point of the congruence, and beside the points of these lines there is no singular point outside SS', for if so, there must be a third self-corresponding plane, in which case the bundles S and S' would be in perspective position† contrary to supposition.

Every line of the congruence generated by S and S' meets both singular lines k_1 and k_2, for any pair of corresponding planes must pass through the same points of these lines; and conversely, every line which meets both these singular lines, the ray SS' included, is a line of the congruence. Those lines belonging to this congruence which pass through any point of k_1 lie therefore in a plane with k_2; and similarly, the lines of the congruence which pass through any point of k_2 lie in a plane with k_1. Out of any point, therefore, of either k_1 or k_2 the congruence is projected by means of a sheaf of planes of the first order, while out of any two points in the line SS' it is projected by two collinear bundles, of which SS' is a self-corresponding ray. The centres of the bundles generating the congruence may, consequently, be any two points of this ray, excepting only the points O_1 and O_2 in which the ray intersects k_1 and k_2 respectively. Out of these two points the congruence is projected in the same manner as out of any other points of the singular lines.

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*Reye, G. d. L., II, S. 86.

†Reye, G. d. L., II, S. 15.

The two singular lines k_1 and k_2 intersect SS', but not in the same point, for then would all lines of the congruence lie in one plane and the bundles would again be in perspective position.

The congruence is as before of the first order, but in this case, of the first class, since an arbitrary plane contains only one line of the congruence, namely, the line joining the points in which k_1 and k_2 respectively intersect this plane.

If now the self-corresponding planes α_1 and α_2 approach indefinitely near to each other and finally coincide in a single plane α_{12}, the lines of singular points k_1 and k_2 will at the same time approach indefinitely near to each other, and will finally coincide in the line k_{12} which lies in α_{12}. Every point of k_{12} is a singular point of the congruence, and besides the points of this line there is no singular point outside SS'. Every line of the congruence meets k_{12}, while those which pass through any one point of the singular line lie in a plane with this line, since they must lie in a plane which passes through all the singular points of the system. The points of the singular line k_{12}, and the planes through this line, are projectively related to one another, each point corresponding to that plane in which lie the lines of the congruence passing through it, for the range of points and the sheaf of planes are both perspective to every regulus of the second order which belongs to the congruence.

Finally, if the self-corresponding planes become imaginary, no real singular point exists outside the line SS', while out of any two points of this line the congruence is projected by means of two collinear bundles as before.

§2.—*The most General Ruled Surface of the Fourth Order.*

4. In the two collinear bundles S and S', respectively, which in the most general case generate a line congruence of the first order and third class, whose singular curve is a non-degenerate twisted cubic k^3, select any pair of corresponding sheaves of planes of the second order, Φ_1 and Φ_1'. These are related projectively to one another and generate by the intersections of corresponding planes a ruled surface, which shall be denoted by F_1^4, each generator of which is a chord of k^3. This ruled surface is of the fourth order, i. e. an arbitrary straight line meets in general four generators of this surface, and if any straight line meet more than four generators it must meet every generator and lie wholly on the surface.

Let p be an arbitrary straight line and π an arbitrary plane through it, cutting the sheaves Φ_1 and Φ_1' in two projective sheaves of lines of the second order, Π and Π'. The points where corresponding rays of Π and Π' intersect are the points in which the plane π cuts the generators of F_1^4. We shall first show that either at most four of these points of intersection, or else an infinite number of them, lie on the arbitrary straight line p.

5. Let a, b, c, d, e be any five rays of the sheaf Π, and a', b', c', d', e' be the corresponding rays of the sheaf Π'. Suppose that the points of intersection A, B, C, D, E of these five pairs of corresponding rays all lie in the straight line p. Project the two sheaves of lines, Π and Π', out of any point O not lying in the plane π, by means of two projective and concentric sheaves of planes of the second order, Ω and Ω', and out of the same point project the line p by means of the plane π_1. The planes α, β, γ, δ, ε of the sheaf Ω, in which lie the lines a, b, c, d, e of the sheaf Π, will intersect the corresponding planes α', β', γ', δ', ε' of the sheaf Ω' in rays a_1, b_1, c_1, d_1, e_1 which lie in the plane π_1 and pass through the point O. Construct a regulus of the second order perspective to the sheaf Ω.* This is cut by the plane π_1 in a conic which is perspective to the sheaf Ω, and consequently projectively related to the sheaf Ω'. But since, by supposition, five pairs of corresponding planes of the two sheaves Ω and Ω' intersect in rays lying in the plane π_1, five points of the conic must lie in the planes corresponding to them of the sheaf Ω', to which it is projective, and therefore the conic is perspective also to the sheaf Ω'.† Hence all pairs of

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* Reye, G. d. L., I, S. 132.

† Reye, G. d. L., I, S. 136.

corresponding planes in the two sheaves intersect in lines which lie in the plane π_1 and pass through the point O, since each pair has in common the point O and a point of the conic. In the plane π_1, then, lie an infinite number of these lines of intersection, and consequently in the line p lie an infinite number of points of intersection of corresponding rays of the two sheaves Π and Π'. The line p, therefore, meets an infinite number, i. e. all of the generators of the surface F_1^4, if in any case it meet more than four of them.

6. Next, let r be an arbitrary straight line not meeting the twisted cubic k^3, and not a chord of k^3. Project the line r out of S by means of the plane ϕ, and find ϕ' the corresponding plane of the bundle S'. Also project r out of S' by means of the plane θ', and find θ the corresponding plane of the bundle S. Then ϕ and ϕ' intersect in a chord of k^3 which meets the arbitrary line r, as do also θ and θ'. In r choose at will three points L, M, N. Project these out of S by the rays l, m, n, and find in the bundle S' the corresponding rays l', m', n'. Similarly, project L, M, N out of S' by the rays l', m', n', and find in the bundle S the corresponding rays l_1, m_1, n_1. The plane λ of the bundle S, determined by the rays l and l_1, corresponds to the plane λ' of the bundle S', determined by the rays l' and l_1'. So also the plane μ, determined by m and m_1, corresponds to the plane μ', determined by m' and m_1', and the plane ν, determined by n and n_1, corresponds to the plane ν' determined by n' and n_1'. The corresponding planes λ and λ' intersect in a chord of k^3 which meets the line r, since both planes pass through the point L of r. Similarly μ and μ', and likewise ν and ν', intersect in chords of k^3 which meet the arbitrary straight line r.

Thus we have obtained five planes in each bundle, viz. ϕ, θ, λ, μ, ν in S, and ϕ', θ', λ', μ', ν' in S', which correspond two and two, corresponding pairs intersecting in chords of k^3 which meet the arbitrary straight line r. These two sets of five planes each determine in the bundles S and S' two corresponding and hence projective sheaves of planes of the second order, Φ_2 and Φ_2', such that the lines of intersection of five pairs of corresponding planes intercept the straight line r. But we have shown that if any straight line meet more than four such lines of intersection, it must meet every line of intersection of pairs of corresponding planes of the two sheaves. Moreover, through every point of r passes a pair of corresponding planes of these sheaves. For, since r lies in the plane ϕ and also in the plane θ', it is cut by the planes of the two sheaves Φ_2 and Φ_2' in two projective ranges of points which lie in one another, and have three self-corresponding points, namely, the points L, M and N. Hence every point of the line is a self-corresponding point, i. e. through every point of r goes a pair of corresponding planes of Φ_2 and Φ_2'. Since no two chords of the twisted cubic k^3 can intersect in a point of r, every chord of k^3 which meets the arbitrary line r must lie in a pair of corresponding planes Φ_2 and Φ_2'. Therefore

All the chords of a twisted cubic curve which meet an arbitrary straight line having no point in common with the cubic, are projected out of any two points of the curve by means of projective sheaves of planes of the second order.

The arbitrarily chosen sheaves Φ_1 and Φ_1' may have, and in general will have, four planes in common with the sheaves Φ_2 and Φ_2' respectively without coinciding with them. Hence the ruled surface F_1^4 may have, and in general will have, four generators which meet the arbitrary straight line r.

Thus we have shown that the ruled surface which is generated by two projective sheaves of planes of the second order is such that an arbitrary straight line may, and in general will, meet four of its generators, and if any straight line meet more than four generators it must meet every generator, and lies wholly on the surface. The surface is therefore of the fourth order.

7. The sheaves of planes Φ_1 and Φ_1' which generate the surface F_1^4 envelope cones of the second order, which we shall denote by K_1 and K_1, respectively, whose vertices are points of the twisted cubic k^3. Each generator of F_1^4 is tangent to both the cones K_1 and K_1', but since either of these cones is determined as soon as the sheaf which envelopes the other is chosen, the surface F_1^4 is completely defined as "that surface whose generators are chords of a twisted cubic k^3, and tangents to a cone of the second order K_1 whose vertex is a point of the cubic." The surface thus defined we shall consider to be the most general Ruled Surface of the Fourth Order.

8. Since the system of tangents to the curve k^3 is projected out of S and S' by means of two projective sheaves of planes of the second order, it forms the system of generators of a ruled surface of the fourth order F^4, which, however, is a developable surface or torse, since each generator intersects its adjacent generators in points of the twisted cubic. The sheaves Φ and Φ' which generate the torse F^4 envelope the cones K and K' which project the points of the cubic out of S and S' respectively.

9. The projective sheaves of planes Φ_1 and Φ_1' which are chosen arbitrarily in the bundles S and S', have in general, and at most four planes in common with the sheaves Φ and Φ' respectively, which project the tangents to k^3, and consequently the surface F_1^4 has in general four generators which are tangents to k^3. Also the cones K_1 and K_1' have in general, and at most, four rays in common with the cones K and K' respectively. If any ray of K_1 coincide with a ray of K, its corresponding ray in K_1' will coincide with the corresponding ray of K', and hence these rays will intersect in a point of k^3. If any ray of K_1 lie either within or without K, its corresponding ray in K_1' will lie within or without K', but these corresponding rays will not intersect.

All planes of Φ_1 and Φ_1' intersect the twisted cubic in the points S and S', respectively, but beside this, if any ray of K_1 lie within the cone K, the plane of Φ_1 through this ray will contain two rays of K, and consequently will meet the cubic k^3 in two points different from S. The corresponding plane of Φ_1' will pass through these same two points, and hence these planes will intersect in a generator of the surface F_1^4 which is an actual chord of k^3. If the sheaves Φ_1 and Φ_1' be so chosen that the cones K_1 and K_1' enveloped by them lie wholly within the cones K and K' respectively, every generator of the surface F_1^4 will be an actual chord of the twisted cubic.

The plane of Φ_1 passing through a ray of K_1 which coincides with a ray of K in general contains a second ray of K, and hence will intersect the corresponding plane of Φ_1' in a generator of F_1^4 which is an actual chord of k^3. The two rays of K which lie in this plane may, however, coincide, as when the two cones are tangent to each other along this ray, in which case the generator becomes a tangent to k^3.

If the sheaf of planes Φ_1 be so chosen that some of the rays of the cone K_1 lie outside the cone K, while others lie inside this cone, there must be at least two rays of K_1, say a and b, which coincide with rays of K. There will be at the same time at least two planes of the sheaf Φ_1, say μ and ν, which coincide with planes of Φ. These planes μ and ν separate the planes of Φ_1 into two groups, in one of which lie the planes through the rays a and b. Every plane of this group will contain two rays of the cone K, and consequently will, with its corresponding plane in Φ_1', give rise to a generator of F_1^4 which is an actual chord of the twisted cubic. Among the planes of the other group there may, and in general will, arise two other planes which are common to the sheaves Φ_1 and Φ.

If these do not appear, the planes of this group will, with their corresponding planes in Φ_1', give rise to generators of F_1^4 which are ideal chords of k^3, since they meet the cubic only in the point S.

If the four common planes of Φ_1 and Φ all appear, they divide the planes of the sheaf Φ_1 into four groups; the planes in two of these groups, with their corresponding planes in Φ_1', give rise to generators which are actual chords of k^3, while the planes of the remaining two groups give rise to generators which are ideal chords of the cubic. The groups which give rise to actual-chord generators alternate between the groups which give rise to ideal-chord generators. Consequently the generators themselves are in general arranged in four groups, of which the four tangent generators already referred to form the boundaries, the two groups of actual-chord generators alternating between the two groups of ideal-chord generators.

10. On the other hand, since the cone K on which the twisted cubic lies has in general four rays in common with the cone K_1, the curve k^3 will in general meet the cone K_1 in four points different from S, and will lie partly within and partly without this cone. Any point of k^3 which lies outside the cone K_1 lies also outside the corresponding cone K_1', and similar conclusions are true for points of the curve which lie either on or inside the cone K_1. Through every point of k^3 which lies outside the cone K_1 will pass two real distinct planes of the sheaf Φ_1, and likewise the corresponding two planes of the sheaf Φ_1', so that, through every point of k^3 which lies outside the cone K_1 will pass two real, distinct generators of the surface F_1^4. Through those points of k^3 which lie inside K_1 will pass no real plane of Φ_1, and consequently no real generator of F_1^4; while through each of the points in which k^3 meets the cone pass two consecutive planes of Φ_1, and consequently two consecutive generators of F_1^4. In other words, through any point of k^3 which lies outside K_1 will pass two real, distinct generators of the surface F_1^4. As this point moves along the curve toward the cone, the two generators tend toward each other and finally coincide for the point of the curve which lies on the cone; as the point moves inside the cone, the generators through it become imaginary. The twisted cubic is thus a double curve on the surface, and may lie either *actually* or *ideally* upon the surface.

11. Since the tangent plane to the surface at any point contains the generator through that point, and since the tangent plane at any point along the

twisted cubic contains the tangent to the cubic at that point, there are at each point of the twisted cubic two tangent planes to the surface, namely, those planes determined by the tangent to the cubic and each of the two generators which meet the cubic at this point. If one of the generators at a point of the cubic coincide with the tangent at this point, the tangent plane through this generator becomes the osculating plane to the cubic, while the other tangent plane in general remains distinct; that is, one of the tangent planes to the surface at a point where a tangent generator meets the double cubic is the osculating plane to the cubic at that point.

At those points of the double curve in which the two generators coincide, namely, at those points of k^3 which lie on the cones K_1 and K_1', the two tangent planes to the surface must also coincide. Such points have been designated by Professor Cayley as *pinch points*, so that the ruled surface of the fourth order F_1^4 has in general four pinch points. These are distributed along the double curve in such a manner as to separate the segments of the curve which lie actually upon the surface from those segments which lie ideally upon the surface. For convenience, these latter segments may be referred to as the *isolated* segments of the double curve. Thus, suppose A, B, C, D be the four points taken in order along the double cubic in which this curve meets the cones K_1 and K_1', and that the segment AB of the curve lies outside the cones. Then the segment BC will lie inside the cones, CD outside, and DA again inside. Through every point of the segments AB and CD will pass two real, distinct generators of the surface; these segments will lie actually upon the surface, and at each point of them there will be two real and distinct tangent planes to the surface. The segments BC and DA of the double curve are isolated segments, and the tangent planes at the points of these segments are imaginary. Through each of the boundary points, A, B, C, D, there pass two consecutive generators, and consequently the two tangent planes at these points coincide. These are, therefore, the four pinch points on the surface.

The four tangent generators which in general appear evidently meet k^3 in those segments which lie actually upon the surface, two in each segment. Denote these generators by r, s, t, v, and suppose that r and s meet k^3 in the segment AB, while t and v meet the curve in the segment CD. If, now, the pinch points A and B move toward each other along the curve and finally coincide at a point P, that is, if the cones K_1 and K become tangent to each other along the ray SP, and likewise the cones K_1' and K' become tangent to each other along the ray $S'P$, then the two tangent generators r and s fall together and become a single tangent generator at the pinch point P; and generally, whenever two pinch points fall together, the generator through this point becomes a tangent generator.

12. As we consider the distribution of pinch points and tangent generators along the double curve, that is, as we consider the possible ways in which the sheaves Φ_1 and Φ_1', and the cones K_1 and K_1' enveloped by them, may be chosen as related to the cones K and K', several different varieties or subforms of the general surface F_1^4 arise. These may be conveniently arranged as follows:

I.—*Subforms in which the Four Pinch Points are all Real.*

(1). The pinch points A, B, C, D are all distinct; the tangent generators r, s, t, v appear on the surface and are also distinct.

In this case the cones K_1 and K_1' have four rays in common with the cones K and K', respectively, namely, those rays which project the points A, B, C, D out of S and S', respectively. The sheaves Φ_1 and Φ_1' likewise have four planes in common with the sheaves Φ and Φ', respectively, these giving rise to the tangent generators r, s, t, v. The cubic curve k^3 lies actually upon the surface throughout two segments, AB and CD, but the remaining two segments are isolated. There are two groups of actual-chord generators and two groups of ideal-chord generators. Each actual-chord generator meets the cubic twice in the same segment.

(2). The pinch points A and B coincide at a point P; the tangent generators r and s fall together and become a single tangent generator at P. The remaining two pinch points C and D, and the tangent generators t and v, exist and remain distinct.

The cones K and K_1 have two distinct rays in common, and are tangent to each other along a third ray SP, while the sheaves Φ and Φ_1 have three planes in common, one of which passes through the ray SP. Similar relations exist between the corresponding cones and sheaves in the bundle S'. The cubic curve lies actually on the surface only throughout the segment CD and at the point P. One group of actual-chord generators reduces to the single tangent generator at the pinch point P.

(3). The pinch points A and B coincide at a point P; also C and D coincide at a point Q. The tangent generators r and s fall together and pass through P, while t and v also fall together and pass through Q.

This case arises when the cone K lies within the cone K_1, the two being tangent to each other along the two rays SP and SQ. The cubic curve is wholly isolated except at the pinch points P and Q. The two groups of actual-chord generators reduce to the single tangent generators through the pinch points P and Q, respectively, all other generators being ideal chords of k^3.

(4). The pinch points B and C coincide at a point Q; the tangent generators s and t also fall together and become a single tangent generator at Q. The remaining pinch points A and D, and the tangent generators r and v, exist and remain distinct.

The cubic curve lies actually on the surface throughout the double segment AQD. One group of ideal-chord generators disappears, and the two groups of actual-chord generators are only separated by the tangent generator through the pinch point Q. One tangent generator, r, meets the cubic in a point of the segment AQ, while the remaining one, v, meets the cubic in the segment QD. Each actual-chord generator meets the cubic twice in the same segment.

(5). The four pinch points are all distinct, but the tangent generators all disappear or are imaginary.

The cones K and K_1 have four common rays, but the sheaves Φ and Φ_1 have no planes in common. The cubic curve lies actually on the surface in two segments, AB and CD, and is isolated in the remaining two segments. Every generator is an actual chord of the cubic, and meets the cubic once in a point of the segment AB and once in a point of the segment CD.

(6). Two pinch points, B and C, of subform (5) coincide at a point Q, at which point there arises a tangent generator. The remaining two pinch points, A and D, are distinct, but there are no other real tangent generators.

The cones K and K_1 are tangent to each other along the ray SQ, and intersect along the rays SA and SD. The sheaves Φ and Φ_1 have only one plane in common, namely, the plane through the common ray SQ. The cubic curve lies actually on the surface throughout the double segment AQD. Every generator of the surface, beside the tangent generator, is an actual chord of the cubic, and meets this curve once in the segment AQ and once in the segment QD.

(7). The pinch points B and C coincide at a point P, while D and A coincide at a point Q. Two tangent generators appear on the surface and pass through the pinch points P and Q, respectively.

In this case the cones K and K_1 are tangent to each other along the rays SP and SQ, while through these rays pass planes common to the two sheaves Φ and Φ_1. The cubic curve lies actually on the surface throughout, and all generators besides the two tangent generators are actual chords of the cubic.

This variety admits of a subdivision according as it arises from the coincidence of pairs of pinch points in subform (1) or in subform (5); that is, according as a generator meets the double curve twice in the same segment PQ, or once in each segment.

(8). Three of the pinch points, A, B and C, coincide at a point P, while the fourth, D, remains distinct. The three tangent generators, r, s and t, fall together and become a single tangent generator through the pinch point P; the fourth tangent generator, v, must appear on the surface and remain distinct.

The cones K and K_1 have two rays SP and SD in common, and have contact of the second order along SP. The sheaves Φ and Φ_1 likewise have two distinct planes in common, of which one is the plane through the common ray SP. The cubic curve lies actually on the surface throughout one of its segments PD, but is isolated in the other segment. There will be one group of actual-chord generators and one group of ideal-chord generators, these groups being separated by the two tangent generators.

(9) and (10). The four pinch points coincide at a point P; the four tangent generators also coincide and pass through P.

The cones K and K_1 have only one ray in common, the ray SP, but have contact of the third order along this ray. The sheaves Φ and Φ_1 have a common plane through SP. Two cases arise according as the cone K_1 lies inside K, or K inside K_1. In the first case (9) the cubic curve lies actually on the surface throughout, and every generator beside the tangent generator at P is an actual chord of k^3. In the second case (10) the cubic curve lies actually on the surface only at the pinch point P, while all generators except the tangent generator are ideal chords of the cubic.

II.—*Subforms in which Two of the Pinch Points are Real and Two Imaginary.*

(11). Any two adjacent pinch points, A and B, and the corresponding tangent generators, r and s, disappear or become imaginary, while the remaining two pinch points, C and D, and the remaining tangent generators, t and v, appear on the surface and are distinct.

The cones K and K_1 have two rays in common, the sheaves Φ and Φ_1 have two planes in common. The cubic curve lies actually on the surface at every point of the one segment CD, and is isolated throughout the other segment. There is one group of actual-chord generators in which the two tangent generators appear, and one group of ideal-chord generators.

(12) and (13). Two pinch points and two tangent generators disappear; the remaining two pinch points coincide at a point P, while the remaining two tangent generators fall together and become a single tangent generator at the pinch point P.

The cones K and K_1 have only one ray in common, namely, the ray SP, along which they are tangent to each other. The sheaves Φ and Φ_1 have but one common plane, the plane through the ray SP. Two different cases arise again according as the cone K_1 lies within the cone K, or the cone K within the cone K_1. In the first case (12) the cubic curve lies actually on the surface throughout, and all generators are actual chords except the tangent generator at the pinch point. In the second case (13) the cubic lies actually on the surface only at the pinch point P, and all generators except the tangent generator are ideal chords of the cubic.

(14). Two pinch points coincide at a point P, the remaining two pinch points disappear. Two tangent generators fall together and pass through the pinch point P, the remaining two appear on the surface and are distinct.

Each of the cones K and K_1 lies outside the other, but they are tangent to one another along the ray SP. The cubic lies actually on the surface throughout, but does not meet all the generators. There is one group of actual-chord generators in which the tangent generator through the pinch point appears, and one group of ideal-chord generators, these groups being separated by the two remaining tangent generators.

III.—*Subforms in which the Four Pinch Points are all Imaginary.*

(15). The four pinch points disappear, but the four tangent generators all exist on the surface and are distinct.

The cones K and K_1 have no ray in common, and each lies wholly outside the other. Of the sheaves Φ and Φ_1, however, there are four common planes which, together with their corresponding planes in Φ' and Φ_1', give rise to the four tangent generators. The cubic curve lies wholly on the surface, but does not meet all the generators. There are two groups of actual-chord generators and two groups of ideal-chord generators occurring alternately, each group of the one sort being separated from the two groups of the other sort by two tangent generators. The two generators which pass through any one point of the cubic always belong to different groups of actual-chord generators.

(16) and (17). The four tangent generators as well as the four pinch points, disappear.

Neither the cones K and K_1, nor the sheaves of planes Φ and Φ_1, have elements in common. Of the cones K and K_1, the one lies wholly inside the other. Two subforms of the surface arise according as K_1 lies inside K, or K inside K_1. In the first case (16) the double curve lies actually on the surface at every point, and all generators are actual chords of the cubic. In the second case (17) the double curve is isolated throughout, and all generators are ideal chords of the cubic.

13. It has already been shown that an arbitrary straight line meets in general, and at most, four generators of the surface F_1^4, consequently an arbitrary plane α cuts the surface in a curve which possesses this same property of being met by an arbitrary straight line of its plane generally and at most in four points. Since α intercepts the double curve of the surface in general in three points, through each of which two generators pass, the curve of section in general has three double points; that is, the surface F_1^4 is cut by an arbitrary plane in general in a curve of the fourth order with three double points.

But this curve is generated by the two projective sheaves of lines of the second order in which the sheaves of planes Φ_1 and Φ_1' are cut by the arbitrary plane α. Hence *two projective sheaves of lines of the second order which lie in the same plane, and which in general have no self-corresponding ray, generate a curve of the fourth order which has three double points.*

14. If the plane of section α contain one generator of the surface, the two projective sheaves of lines in which Φ_1 and Φ_1' are cut will have this generator as a self-corresponding ray, and will generate, in addition to this straight line, a curve which is intercepted by this generator as well as every other straight line of the plane generally and at most in three points. This curve must have one double point, since the plane α cuts the double curve of the surface in one point other than the intersections of the generator with the double curve; that is, any section of the surface F_1^4 by a plane through one generator consists of this generator and a curve of the third order with a double point.

If the generator p through which the plane is passed be an actual chord of the double curve k^3, in each point of intersection of this generator with k^3 the plane will intercept a second generator, and consequently the curve of section will always pass through these two points. These points remain fixed for all planes through the generator. The third point of intersection of the generator with the curve of section is that point at which the plane cuts the two consecutive generators. The plane is said to be tangent to the surface at this point, for it not only contains the generator through the point but also a second line which here meets the surface in three consecutive points. Every plane through a generator is tangent to the surface at some point along this generator, since at some point it cuts the two consecutive generators; the point of tangency varies along the generator with each different plane of section.

When the point of tangency of the plane coincides with either of the intersections of the generator p with k^3, this point must be the double point on the cubic curve of section, since at this point the tangent plane cuts not only the generators consecutive to p, but also the second generator through this point of the double curve. The tangent to the double curve at this point also lies in this plane, as has already been stated.

If the actual-chord generator lying in a plane of section pass through one of the four pinch points of the surface, the curve of section must be tangent to this generator at the pinch point. In this point the plane cuts the generators consecutive to the given generator, and both on the same side, while the double point on the curve of section exists elsewhere. In a certain sense every plane through this generator is tangent to the surface at the pinch point, for at this point every plane cuts the consecutive generators. The curve of section of the plane containing the generator and the tangent to k^3 at the pinch point consists of a cubic with a cusp at the pinch point, since for this plane the double point of the cubic falls into the pinch point. This is the true tangent plane at this point.

If, however, a plane be passed through an ideal-chord generator, the curve of section will meet this generator only in the point at which the plane is tangent to the surface; for, if it meet in any other point, the plane must cut a second generator at this point, which is impossible, since no second generator meets an ideal-chord generator.

If, now, a section of the surface be made by an arbitrary plane through one of the four tangent generators, for example, through the generator r which is tangent to the double curve at the point R, the cubic branch of the curve of section will cut this generator in the point at which the plane is tangent to the surface, and will be tangent to this generator at the point R. If the plane touch the surface at R, this point becomes the double point of the cubic curve of section.

15. At every point in which the twisted cubic k^3 lies actually on the surface F_1^4, two real generators of the surface intersect. The curve of section of the plane of these generators consists of these two straight lines and a conic, since the plane cuts the two sheaves Φ_1 and Φ_1' in two projective sheaves of lines of the second order which have two self-corresponding rays, and which generate in addition to these rays a curve of the second order which is perspective to both sheaves of lines.* If the two generators through any point P meet the double curve a second time in the points Q and R, respectively, the conic of section must pass through these two points, since in each of them the plane of section cuts a second generator. The conic must also intersect each of the generators PQ and PR in a second point, at which points the plane of section is tangent

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*Reye, G. d. L., I, S. 137.

to the surface, so that the plane of two generators is a bitangent plane to the surface. Every bitangent plane must contain two generators, and therefore cuts the surface along these generators and in a conic.

If one of the generators, PQ, be a tangent generator, the conic will pass through the point P, while the point of tangency along the generator PR will coincide with P, but the point of tangency along the tangent generator will in general be different from P. Suppose one of the generators through P pass through a pinch point A, then must the conic be tangent to the generator PA at A. If, finally, the bitangent plane contain the two consecutive generators through a pinch point P, these generators meeting the double curve a second time at Q, then it also contains the tangent to k^3 at Q, and the remaining part of the curve of section is a conic passing through Q. This plane is tangent to the surface all along the generator PQ except at the pinch point, the generator therefore being *torsal.* Through each point of the double curve k^3 there pass in general three distinct bitangent planes.

16. Let the two conics arising from the section of the surface F_1^4 by two arbitrary bitangent planes, σ and σ', be denoted by f_1 and f_1', respectively. These are related projectively to one another, since each is perspective to both sheaves of planes which generate F_1^4, and determine completely a collinear relation between the planes in which they lie, and in such a manner that these planes have no self-corresponding point. For each point of the conic f_1 corresponds to one and but one point of the conic f_1', namely, those points which lie in the same generator of F_1^4 are corresponding points. Whenever two generators of the surface intersect, the four points in the conics f_1 and f_1' arising from these must lie in one plane, the bitangent plane through the two generators. If, now, the planes σ and σ' have a self-corresponding point C, any ray of σ which passes through C and cuts the conic f_1 in two points, P and Q, corresponds to a ray of σ' which also passes through C, and which cuts the conic f_1' in two points P' and Q', corresponding to P and Q respectively. The generators PP' and QQ' consequently lie in a plane determined by a pair of corresponding rays through C, and similarly, each pair of generators which intersect lies in a plane which passes through C; that is, all the bitangent planes of the surface F_1^4 pass through one point, which is evidently impossible. Hence the planes σ and σ' have no self-corresponding point, and therefore generate a sheaf of planes of the

third order to which the bitangent planes of the surface F_1^4 belong, since each bitangent plane intersects σ and σ' in a pair of corresponding lines.*

17. The generators of the surface F_1^4 are not only the lines of intersection of corresponding planes in the sheaves of the second order Φ_1 and Φ_1', but are at the same time lines joining corresponding points in the projective conics f_1 and f_1', so that the surface may be generated as well by the projective conics as by the projective sheaves of planes. And since the collinear planes σ and σ', in which f_1 and f_1' are corresponding figures, have no self-corresponding element, the method of generating the surface by means of the projective conics is the exact reciprocal of the method by which the surface was originally generated; that is, by means of the projective sheaves of planes. Hence the surface F_1^4 is its own reciprocal.†

18. It has been already shown that the conic in which the surface F_1^4 is cut by an arbitrary bitangent plane intersects the double curve of the surface in two points, through which pass the two generators lying in the plane of the conic, one generator through each point, and two other generators which are cut in these points by the bitangent plane. Through every other point of this conic there passes one generator of the surface, and consequently no chord of the twisted cubic which is not a generator of the surface. From this it follows immediately that *those chords of the twisted cubic curve k^3 which intercept a conic having two points in common with the cubic, form the system of generators of the most general ruled surface of the fourth order.* Through each of the points of intersection of the conic with the cubic there passes an infinite number of chords which project the points of the cubic, and which consequently lie on cones of the second order. Only two of the rays of each of these cones, however, belong to the surface F_1^4.

19. That the chords of the twisted cubic curve k^3 which meet an arbitrary conic r^2 having two points, A and B, but no third point in common with the

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* Reye, G. d. L., II, S. 94.

† This proof that the surface F_1^4 is of the same form as its reciprocal proceeds upon the supposition that two real bitangent planes to the surface exist. But this is not the case in some of the subforms, as for example in the subform (17). However, if we study the reciprocal method of generating the surface, it is readily seen that the generating conics may be so chosen in the collinear planes that these same subforms arise.

cubic form the system of generators of a ruled surface of the fourth order, may be shown directly in the following manner:

Out of any point S of k^3, not lying in the plane of r^2, project the points of this conic by means of the rays of a cone of the second order, P_1, and find in the bundle S', whose centre is likewise a point of k^3 not lying in the plane of r^2, the corresponding cone P_1'. Two pairs of corresponding rays of these cones will intersect in the points A and B, namely, SA and $S'A$ are corresponding rays, SB and $S'B$ are corresponding rays. Also, out of S' project the points of r^2 by means of the rays of a cone of the second order, P_2', and find in the bundle S the corresponding cone P_2. The rays SA and $S'A$, SB and $S'B$ belong as well to these cones, P_2 and P_2', as to the cones P_1 and P_1'. Thus we have in the bundle S two cones P_1 and P_2 which have two common rays SA and SB, and in the bundle S' the two cones P_1' and P_2' which correspond to these, and which also have two common rays, $S'A$ and $S'B$.

Now the two cones P_1 and P_2' are both perspective to the conic r^2, and consequently we may establish a projective relation between them by correlating to one another those rays which pass through the same points of r^2. But the rays of the cones P_1 and P_1', as also the rays of P_2 and P_2', are related projectively to one another by virtue of their correspondence in the two bundles. By this means is established a projective relation between the rays of the cones P_1 and P_2, as also between the rays of P_1' and P_2', and in such a manner that the rays SA and SB are self-corresponding rays of the cones P_1 and P_2, and $S'A$ and $S'B$ are self-corresponding rays of the cones P_1' and P_2'.

The plane of the bundle S which is determined by two corresponding rays of the cones P_1 and P_2 corresponds to the plane of the bundle S' which is determined by the rays corresponding to these, and which also correspond to one another, in the cones P_1' and P_2'. Therefore, these two planes intersect in a chord of k^3, and this chord must meet the conic r^2, since the two planes pass through the same point of r^2. For example, let E be any point of the conic r^2, e_1 the ray of P_1 which passes through E, and e_1' the corresponding ray of P_1'; also, let e_2' be the ray of P_2' which passes through E, and e_2 the corresponding ray of P_2. Then will e_1 and e_2 be corresponding rays of the cones P_1 and P_2, e_1' and e_2' will be the rays corresponding to these, respectively, and to one another in the cones P_1' and P_2', and the plane of the bundle S determined by the rays e_1 and e_2 will correspond to the plane of the bundle S' determined by the rays

e_1' and e_2'. These planes will intersect in a chord of k^3 which passes through E, for in each plane there lies a ray which passes through that point.

Since E is any point of the conic r^2, through every point of this conic there passes a chord of k^3 which is projected out of S by a plane containing a pair of corresponding rays of the cones P_1 and P_2, and out of S' by a plane which contains the rays of P_1' and P_2' corresponding to these, and to one another. No chord of the cubic can meet the conic r^2 which is not so projected out of S and S', since in no point of the conic can two chords intersect, the points A and B being exceptional points, through each of which there passes an infinite number of chords which lie, in each case, on a cone of the second order projecting the cubic.

Let π be an arbitrary plane which cuts the cones P_1 and P_2, P_1' and P_2' in the conics p_1^2 and p_2^2, $p_1'^2$ and $p_2'^2$, respectively. Of these conics, p_1^2 and p_2^2 are related projectively to one another and have two points of intersection as self-corresponding points, namely, those points in which the two self-corresponding rays SA and SB of the cones are cut by the plane π. The system of straight lines joining pairs of corresponding points in these two conics forms a sheaf of lines of the second order perspective to both conics,* and hence the planes of the bundle S determined by pairs of corresponding rays of the cones P_1 and P_2 form a sheaf of planes of the second order, Φ_{12}, perspective to both cones. Similarly, the conics $p_1'^2$ and $p_2'^2$ generate a sheaf of lines of the second, and the cones P_1' and P_2', of which these conics are sections, generate in the bundle S' a sheaf of planes of the second order, Φ_{12}', perspective to both cones P_1' and P_2', and therefore projectively related to the sheaf of planes Φ_{12}. Corresponding planes in these two sheaves pass through the same point of r^2, and conversely, through every point of r^2 goes a pair of corresponding planes of these sheaves. Hence the sheaves of planes Φ_{12} and Φ_{12}' generate a ruled surface of the fourth order, each of whose generators is a chord of the cubic k^3 which meets the conic r^2, while through every point of r^2 there passes a generator of this surface.

The conic r^2 must lie entirely outside the cones K_{12} and K_{12}', which are enveloped by the sheaves of planes Φ_{12} and Φ_{12}', respectively, otherwise it would not be possible for a generator of the surface arising from these sheaves to pass through every point of r^2. Two planes of each sheaf pass through each point of r^2, only one pair of which are corresponding planes, however, except in the points

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* Reye, G. d. L., I, S. 137.

A and B, where both pairs are corresponding. The twisted cubic k^3 must lie actually upon this surface, at least in the points A and B and in the third point of intersection of the plane of the conic r^2 with the cubic, since through this third point two chords of the cubic pass which intercept the conic in A and B, respectively, and in two other points.

§3.—*The Surface with a Double Twisted Cubic and a Single Straight Line Director.*

20. It was shown in Art. 6 that all the chords of the twisted cubic k^3 which meet an arbitrary straight line r having no point in common with the cubic, are projected out of any two points of this curve, S and S', by means of two projective sheaves of planes of the second order; consequently these chords of the cubic form the system of generators of a ruled surface of the fourth order. Let this surface be denoted by F_2^4. Since an arbitrary plane through the straight line r cuts the curve k^3 in general in three distinct points, and contains three chords of this curve, the generators of the surface F_2^4 in general lie by threes in planes through the straight line r. The twisted cubic k^3 is, as in the most general case, a double curve on the surface.

On the other hand, if any plane section of the ruled surface generated by two projective sheaves of planes of the second order, chosen arbitrarily in the collinear bundles S and S', contain three generators of this surface, then must every generator of the surface meet some one straight line lying in this plane. For this plane of section cuts the sheaves generating the surface in two projective sheaves of lines of the second order, which have three self-corresponding rays. The straight line r joining the points of intersection of any two pairs of corresponding rays in these sheaves of lines also meets the three self-corresponding rays, and therefore contains five, and consequently all points of intersection of pairs of corresponding rays. Hence all generators of the surface will meet this one straight line r.

Since common rays of two sheaves of lines of the second order lying in the same plane occur in pairs, and the two sheaves of the preceding paragraph have three self-corresponding rays, there must be some fourth ray, d, which is common to both. This common ray, d, is cut by the remaining rays of the two sheaves in two projective ranges of points which, however, have three self-corresponding points, and which consequently coincide throughout. All pairs of corresponding rays in the two sheaves therefore intersect in the points of the straight line d; but they also intersect in the points of r, which is impossible unless d and r coincide. The common ray d cannot be a self-corresponding ray of the two sheaves, for then would four chords of the twisted cubic lie in one plane. The straight line r, therefore, which is met by every generator of the surface, must be the line of intersection of two non-corresponding planes of the generating sheaves, and is consequently a common ray of those two sheaves of lines in which the generating sheaves of planes are cut by any plane through r. Through every point of r there passes one and only one generator of the surface, since in every point of d, one and only one pair of corresponding rays of the sheaves of lines intersect. The line r is therefore a single director line on the surface, through every point of which there passes a generator. In no case can this line meet the twisted cubic, for if so, through the point of intersection would pass more than one generator of the surface.

Thus it has been shown that *those chords of a twisted cubic curve which meet an arbitrary straight line having no point in common with this curve, form the system of generators of a Ruled Surface of the Fourth Order on which the twisted cubic lies as a double curve, and which is such that an arbitrary plane passed through the straight line director contains in general three generators of the surface; and conversely, if any three generators of a Ruled Surface of the Fourth Order which has a twisted cubic for its double curve lie in one plane, then every generator of that surface will meet one straight line which has no point in common with the cubic.*

21. The sheaves of planes Φ_2 and Φ_2' which generate the surface F_2^4 occupy a wholly special position with respect to the cubic curve k^3. Each of the two generators, AB and AC, which pass through any point A of the double cubic, and which meet this curve a second time in the points B and C, respectively, is tangent to both cones K_2 and K_2' which are enveloped by the sheaves Φ_2 and Φ_2'. In order that the chord BC, the only remaining chord of the cubic to be found in the plane of AB and AC, may also be a generator of the surface F_2^4, in which case, as was shown above, every generator of the surface will meet one straight line in this plane, the sheaves Φ_2 and Φ_2' must be so chosen that this chord, as well as the chords AB and AC, is tangent to the cones K_2 and K_2'; that is, this chord must also lie in a pair of corresponding planes of the sheaves

Φ_2 and Φ_2'. The sheaf of planes Φ_2 is therefore so situated with respect to the cone K which projects the cubic curve out of S, and similarly, the sheaf Φ_2' to the cone K', that three planes of the sheaf intersect, two and two, in three rays of the cone. It is well known that when one such triplet of planes exists in the sheaf Φ_2 there is necessarily an infinite number of such triplets;* consequently, if there be on the surface F_2^4 one set of three generators which lie in the same plane, there will necessarily be an infinite number of such sets, and the planes in which these sets lie must all pass through that line which is met by every generator of the surface.

If there be any ideal-chord generators on the surface F_2^4, these will lie each in one plane through the straight line director. The planes through this director will thus be separated into groups, each of the planes in one group containing three actual-chord generators, those in the next consecutive group containing each one ideal-chord generator, there being at most two groups of each kind, and the groups being separated by planes which contain in general one tangent generator and one actual-chord generator, it being possible, however, for these to coincide as a single tangent generator.

It is readily seen that when two of the generators lying in one plane through the straight line director approach indefinitely near to each other, as in the case of the boundary plane between two groups, these consecutive generators pass through one of the pinch points on the surface, and that the third generator of this plane is tangent to the double curve. In the first place, when two consecutive generators intersect in a point of the double curve this point is always a pinch point, since the two tangent planes to the surface at this point are coincident. But aside from this, if two of the three generators lying in one plane through the straight line director approach indefinitely near to each other, two of the three planes of the sheaf Φ_2 which give rise to this set of generators must also approach indefinitely near to each other, in which case the third plane of this triplet becomes tangent to the cone K. But this can only occur when the two consecutive planes of this triplet pass through a ray common to the cones K and K_2. Similar relations hold for the sheaf Φ_2' and the cones K' and K_2'. Hence the two consecutive generators pass through a pinch point while the third generator of the set to which these belong is a tangent generator.

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* Compare Cremona, Elem. of Proj. Geom., p. 244; Poncelet, Proj. Prop., art. 565.

22. Any plane through the straight line director of this surface F_2^4 is evidently tangent to the surface at every point in which a generator lying in this plane intersects the director, and hence each plane through the straight line director is in general a tritangent plane to the surface. The points of tangency of a tritangent plane vary along the director as the plane rotates about this line. They remain distinct, or two, or all three of them coincide, or two of them become imaginary, according as the generators lying in the plane remain distinct, or two, or all three of them coincide, or two of them become imaginary. Every bitangent plane of the surface F_2^4 is thus a tritangent plane, while the three bitangent planes which in the general surface F_1^4 pass through any one point of the double curve coincide in a single plane through r. The bitangent planes of this surface therefore form a triply counting sheaf of the first order whose axis is the straight line director.

23. Unlike the general surface F_1^4, this species F_2^4 differs in form from its reciprocal. The reciprocal surface which may be denoted by $F_{2'}^4$ must satisfy the conditions that through each point in which two generators intersect a third shall pass; these triple points all lie in one straight line; any plane through this straight line contains but one generator of the surface; the planes containing two generators of the surface, i. e. the bitangent planes, form a sheaf of the third order. This surface $F_{2'}^4$ is thus a triple line surface, and cannot be generated by means of two projective sheaves of planes of the second order.

24. A ruled surface of the fourth order with a non-degenerate twisted cubic for its double curve cannot have more than one straight line director. For, if two, then any plane through one of them contains in general three generators of the surface which meet the second director and which do not pass through one point. Hence the two directors must lie in one plane, and consequently all generators of the surface would lie in one plane, which is evidently impossible.

§4.—*The Surface with a Double Conic and a Double Straight Line which is not a Generator.*

25. Let us next consider the case in which the bundles S and S' have one self-corresponding plane, but no other self-corresponding element. The line congruence arising from these bundles is, as we have seen, of the first order and

second class, while the curve of singular points consists of a conic k^2 and a straight line k which meet in a point O, but which are not in the same plane.

As before, select at will in the bundles S and S' a pair of corresponding sheaves of planes of the second order, Φ_3 and Φ'_3. These are related projectively to each other and generate, as in the general case, a ruled surface of the fourth order, F^4_3. Each generator of the surface is a line of the congruence, and is tangent to both the cones K_3 and K'_3 which are enveloped by the sheaves Φ_3 and Φ'_3, respectively. But since any line of the congruence which is tangent to the one cone K_3, the lines through the vertex being generally excluded from our consideration, is necessarily tangent to the other cone K'_3, and all such lines of the congruence lie in pairs of corresponding planes of the sheaves Φ_3 and Φ'_3, the surface F^4_3 is completely defined when we say that *those lines of a congruence of the first order and second class which are tangent to a cone of the second order whose vertex is a point of the singular conic of the congruence form the system of generators of a ruled surface of the fourth order.*

Every straight line which meets the conic k^2 and the straight line k is a ray of the congruence, and consequently the surface F^4_3 may be considered as generated by a straight line which constantly meets both k^2 and k, and which moves so as to continually touch a cone of the second order whose vertex is a point of the conic k^2.

26. Through each point of k^2 and of k in general pass two pairs of corresponding planes of the sheaves Φ_3 and Φ'_3, and consequently two generators of the surface F^4_3. Hence the conic k^2 and the straight line k are double curves on the surface, the two generators through any point of these curves being real and distinct, coincident or imaginary according as the point lies without, on or within the cones K_3 and K'_3. These cones in general intercept the double conic k^2 and the double straight line k in four points, two of which are on the conic and two on the straight line. These four points of intersection are, as in the most general case, the pinch points of the surface.

27. The two generators through any point of the nodal conic lie in a plane with the nodal straight line. The generators through O, the point of intersection of k^2 and k, meet neither branch of the double curve elsewhere, and lie in the plane determined by k and the tangent to k^2 at this point O. The point O is

thus a "singular" point on the double curve. The pinch points on each branch of the double curve divide the branch into two segments, of which the one lies actually upon the surface, while the other segment is isolated. If the singular point O appears in that segment of the nodal conic which lies actually upon the surface, it must likewise appear in the segment of the nodal straight line which lies actually upon the surface, since in this case it lies outside the cones K_3 and K_3', and similarly, if the singular point appears in the isolated segment of the one branch, then also in the isolated segment of the other branch. If a pinch point on one branch of the double curve coincide with the singular point O, so also must one of the pinch points on the other branch coincide with this singular point, and beside this point there will always exist one real pinch point on each branch of the double curve.

28. There are then five subforms of the surface F_3^4 to be considered, as follows:

(1). The pinch points are all real and distinct. The cones K_3 and K_3' intercept k^2 and k, each in two distinct points. Each branch of the double curve has one segment which lies actually upon the surface, and one isolated segment. This subform may be further subdivided according as the singular point O lies actually upon the surface, or appears in the isolated segments of the double curve.

(2). The pinch points on the nodal conic are real and distinct, but on the nodal straight line are imaginary. The cones K_3 and K_3' intercept the conic k^2 in two points, but do not intercept the straight line k. The nodal line lies actually upon the surface throughout, the conic has one segment which lies actually upon the surface, and one isolated segment. The singular point O must appear in that segment of the conic which lies actually upon the surface.

(3). The pinch points on the nodal conic are imaginary, but on the nodal straight line are real and distinct. The cones K_3 and K_3' intercept the straight line k in two points, but do not intercept the conic k^2. The nodal conic lies actually upon the surface throughout, while of the nodal straight line there is one isolated segment and one segment which lies actually upon the surface, the singular point O always appearing in this latter segment.

(4). One pinch point on each branch of the double curve coincides with the singular point O; a second pinch point exists on each branch. The cones K_3 and K_3' pass through the singular point O, and consequently intercept both k^2 and k in a second point. There is one isolated segment on each branch of the double curve, the singular point O forming one of the boundaries of each segment.

(5). The pinch points are all imaginary. The cones K_3 and K_3' are so situated as not to intercept either k^2 or k. Both nodal conic and nodal straight line lie actually upon the surface at every point.

29. The section of the surface F_3^4 by an arbitrary plane is, as for the most general surface F_1^4, a quartic curve which has in general three double points. The curve of section by a plane through a single arbitrary generator consists of this generator and a cubic with a double point, and since every generator of this surface meets both the nodal conic and the nodal straight line in real points, the cubic branch of the curve of section must intersect the generator in three real points, namely, in the points P and Q in which the generator meets k^2 and k, respectively, and in a third point R at which the plane is tangent to the surface. When the point of tangency falls in with the point P this point becomes the double point on the cubic, and the plane contains the tangent to the conic at this point. The tangent plane to the surface at any point of the nodal conic is thus determined by the tangent to the conic at that point and a generator passing through the point. At each point of the nodal conic there are in general two distinct tangent planes to the surface. Only at the pinch points and at the singular point O do these two planes coincide, while for points in the isolated segment the tangent planes are imaginary.

When the point of tangency of the plane through a generator PQ coincides with the point Q in which the generator intersects the nodal line k, the plane must contain this nodal line, since it contains three consecutive points of the line. The curve of section in this case consists of the generator PQ, the second generator PQ' through the point P of the conic, and the straight line k counting doubly, since in each point of k two generators in general are intercepted. Every plane through the nodal line k contains in general two generators which intersect in a point of the nodal conic, and is therefore tangent to the surface at two points on k, namely, the points of intersection of these generators with k, these being in general distinct. Moreover, at every point of the nodal line k there are two planes which are tangent to the surface at this point; these may, however, coincide or be imaginary.

30. The plane of the two generators through any point of the nodal conic is thus a bitangent plane to the surface passing through the nodal line k, the totality of such planes forming a sheaf of the first order whose axis is k. Since the two generators passing through a pinch point on the nodal conic coincide, the points of tangency of the plane determined by this doubly counting generator and the nodal line k also coincide and the plane is tangent to the surface all along this generator, which is consequently torsal. The tangent plane at the pinch point, however, is as usual the plane of the generator and the tangent to the conic at this point.

The plane of the two generators through any point Q of the nodal line k cuts the nodal conic in two points P and P' in which the two generators meet the conic. This plane cuts the sheaves Φ_3 and Φ_3' generating the surface in two projective sheaves of lines of the second order which have two self-corresponding rays, and which consequently generate a curve of the second order perspective to both sheaves of planes. This conic is met by every generator of the surface and passes through the two points P and P' of the nodal conic, since in these points generators not lying in the plane of section are intercepted by this plane. The conic of section also intersects the two generators of its plane in P and P', and in two additional points at which the plane cuts the generators consecutive with these, that is, at which the plane is tangent to the surface. Every plane therefore through two generators which intersect in a point of the nodal line k is a bitangent plane, and cuts the surface along these two generators and in a conic which passes through the points of tangency of the plane, and which meets the nodal conic in two points.

The plane of the two coincident generators at a pinch point on the nodal line is tangent to the surface all along this generator except at the pinch point, and contains the tangent to the nodal conic at the point P in which this torsal generator meets the nodal conic. The curve of section of this plane consists of the torsal generator counting doubly, and a conic which meets the nodal conic k^2 in two consecutive points, and which consequently has a common tangent with k^2 at this point. The plane of the two generators through the singular point O cuts the surface along these two generators and along the nodal line k, which as a part of the curve of section counts doubly. The two points of tangency of this plane coincide at the singular point; the two tangent planes at the singular point likewise coincide in this plane. The singular point O is thus in a certain sense a pinch point, but differs from an ordinary pinch point in that the two generators through it do not in general coincide. The curve of section of the plane of the nodal conic consists simply of this conic counting doubly.

31. The bitangent planes to this surface which are determined by pairs of generators intersecting in a point of k form a sheaf of planes of the second order. Let α and α' be any two of these bitangent planes which meet k in the points Q and Q', and contain the pairs of generators a, b and a', b', respectively. Denote the conics of section lying in these planes by a^2 and a'^2, and let the generators a and b meet the conic a^2 of their plane in the points A and B, respectively, at which points the plane α is tangent to the surface; similarly, let the generators a' and b' meet the conic a'^2 of their plane in A_1' and B_1', respectively, at which points α' is tangent to the surface. Now, the generators a and b must meet the conic a'^2 in real points, but this can only occur in the points in which a'^2 cuts the line of intersection, l, of the planes α and α'. Denote these points of a'^2 by A' and B'.

Since the conics a^2 and a'^2 are both perspective to the sheaves Φ_3 and Φ_3', they are projectively related to each other, those being corresponding points which lie on the same generator of the surface. Consequently A and A', B and B' are pairs of corresponding points of these projective conics. The conics determine a collinear relation between the planes α and α' such that the line AB of α corresponds to the line $A'B'$, or l, of α', and these lines are distinct, so that of the planes α and α' there cannot be more than one self-corresponding point. An arbitrary plane through k contains in general two generators of the surface F_3^4, and consequently two pairs of corresponding points of the conics a^2 and a'^2, and therefore this plane cuts α and α' in a pair of corresponding lines, and the lines AB and l in a pair of corresponding points. The point L therefore in which AB and l intersect is a self-corresponding point of the planes α and α'.

Through this self-corresponding point must pass each bitangent plane which contains two generators intersecting in a point of k; for, every such plane cuts α and α' in a pair of corresponding lines which intersect in a point of l, but this is only possible when the plane passes through the self-corresponding point L. The totality of such bitangent planes therefore cuts α and α' in two projective sheaves of lines whose centres coincide at L and which have no self-corresponding ray. Hence these bitangent planes form a sheaf of the second order whose centre is the point L.

Thus the bitangent planes of the surface F_3^4 lie in two distinct sheaves, one of the first order whose axis is k, and the other of the second order.

32. Since every generator of the surface F_3^4 meets the conic of section lying in any bitangent plane, and through every point of this conic there passes a generator, the surface F_3^4 may be defined as being formed from *those rays of a congruence of the first order and second class which meet a conic having two points in common with the singular conic* k^2, *the two points being distinct from the singular point* O; or it may be defined as the surface generated by a straight line moving so as always to meet two conics k^2 and m^2, which lie in different planes and have two points in common, and a straight line k lying in a plane with neither conic but having a point in common with one of them. It is readily shown, conversely, as in the most general case, that those rays of the congruence which intersect a conic having two points in common with k^2 are projected out of S and S' by means of two projective sheaves of planes of the second order, and consequently that these rays form the system of generators of a ruled surface of the fourth order.

33. It has already been noticed that the two conics of section lying in two arbitrary bitangent planes which contain pairs of generators intersecting in points of k are projectively related to each other, and determine by their projectivity the collinearity of the planes in which they lie in such a manner that these planes have one and but one self-corresponding point. The generators of the surface F_3^4, therefore, may be considered as the lines joining pairs of corresponding points in two corresponding conics chosen in collinear plane fields which have one self-corresponding point. But this method of generating the surface F_3^4 is the exact reciprocal of that by which the surface was originally generated. Hence the surface which is reciprocal to F_3^4 is of the same form as F_3^4, or the surface F_3^4 is its own reciprocal.

34. The sheaves Φ_3 and Φ_3' must be so chosen as not to include the self-corresponding plane α of the bundles S and S'. For any line of this plane which passes through the singular point O might then be considered a generator of the surface F_3^4, and the plane would contain an infinite number of generators and would therefore belong wholly to the surface. An arbitrary straight line would always intercept a generator lying in the plane α, and beside this would meet in general three and only three generators of the surface. Hence, if the sheaves Φ_3 and Φ_3' include the self-corresponding plane α, the surface F_3^4 breaks up into this plane and a ruled surface of the third order.

Moreover, if the plane α belong to the sheaves Φ_3 and Φ_3' the singular line k will be a double line on the surface as in the general case, but through each point of the singular conic there will pass only one generator aside from the generators lying in the plane of the conic, that is, only one generator of the cubic surface. The plane γ of any two generators which intersect in a point of the nodal line k will cut the sheaves Φ_3 and Φ_3' in two projective sheaves of lines of the second order which have three self-corresponding rays, namely, the two generators lying in the plane and that line in which γ cuts the plane α. Hence every generator of the surface arising from these sheaves meets some one straight line g lying in the plane γ, which does not meet either k^2 or k. Conversely, if every generator of the surface F_3^4 meet some one straight line g different from the nodal line k, then must the sheaves of planes generating this surface include the self-corresponding plane α. For, through every point of this line there passes a generator, in particular through the point P where g meets the plane α. The ray projecting this point out of S is a ray of α and corresponds to some ray of S' which also lies in α. Similarly, the ray projecting the point P out of S' corresponds to some ray of S lying in α. Hence the generator through P must be the line of intersection of the plane α with itself; that is, the plane α is a self-corresponding plane of the two sheaves generating the surface. Therefore that variety of the surface F_3^4 which has a single straight line director in addition to the nodal conic k^2 and the nodal straight line k degenerates into a ruled surface of the third order and a plane, namely, the plane of the nodal conic.

It will be observed that this latter is the case which arises when the two pinch points on the nodal conic of the surface F_3^4 coincide, for then must the cones K_3 and K_3' be tangent to the conic k^2, and hence the plane of the conic becomes a plane of the sheaves Φ_3 and Φ_3' generating the surface.

§5.—*The Surface with a Double Conic and a Generator which intersects every other Generator.*

35. If now the generating sheaves of the last section be so chosen as to avoid the self-corresponding plane α, but so that one of planes of Φ_3 contains the singular line k, then the corresponding plane of Φ'_3 will also pass through this line, and hence k will be a generator of the surface. This gives rise to what may be considered a distinct variety of the surface which we shall designate by F^4_4, and the sheaves of planes generating it by Φ_4 and Φ'_4, respectively; that is, the ruled surface F^4_4 is of the fourth order, and has a nodal conic k^2 and a nodal straight line k which is also a generator of the surface.

Through each point of the nodal conic there pass in general two distinct generators of the surface which meet the nodal line k. One of these two generators which pass through the singular point O, however, coincides with the line k, while the other remains distinct, and lies in that plane through k which is tangent to k^2. Through each point of the nodal line k there pass two generators of this surface, the line k itself and a second generator which meets the nodal conic. In one single point of this line there is but one generator, the line k itself. In other words, at one single point of the line k the second generator coincides with k; this point is therefore a pinch point of the surface. Thus every plane through k contains in general three generators of the surface, namely, the line k and the two generators through that second point of the nodal conic which lies in the plane

36. The curve of section of the surface F^4_4 by an arbitrary plane through the nodal line k consists of the three generators lying in the plane, the generator k counting doubly, since in the points of this line the remaining generators are cut by the plane of section. The section of the surface by that plane through k which is tangent to the nodal conic consists of the line k counting triply and the second generator through O. This plane is tangent to the surface all along the line k, this line being a torsal generator. At every point along the line k, except at the pinch point and at the singular point O, there is a second tangent plane, namely, the plane of k and the second generator through the point. But every such plane contains also a third generator, and is consequently a tritangent plane to the surface, the points of tangency being the pinch point on k and those two points of k in which the remaining two generators of the plane inter-

sect this line. Every bitangent plane of this surface is therefore a tritangent, the points of tangency all lying on k, and the totality of such planes forming a sheaf of the first order whose axis is k.

37. This variety of the surface arises when the two pinch points of the more general surface F_3^4 which lie in the nodal line k coincide, for then is this line tangent to the cones K_3 and K_3', and hence lies in a pair of corresponding planes of Φ_3 and Φ_3'. The surface F_4^4 has then always one real pinch point on the nodal line k, as we have already seen, and may have either two or none on the nodal conic. A special case arises when one of the pinch points on the nodal conic coincides with the singular point O, in which case the pinch point on the nodal line also coincides with this point.

38. This surface, like the surface F_2^4, differs in form from its reciprocal. The reciprocal surface, which may be denoted by $F_{4'}^4$, must be such that through each point of intersection of two generators a third will pass, and always the same third generator; that is, the surface $F_{4'}^4$ is a triple line surface, the triple line being a generator through each point of which two other distinct generators pass. Each plane through the triple line contains but one other generator.

§6.—*The Surface with two Double Straight Line Directors and a Double Generator.*

39. Let us next consider the case in which the ray SS' of the bundles S and S' is a self-corresponding ray. As has been already observed (Art. 7), these bundles generate a congruence of the first order and first class, whose singular points lie on two straight lines k_1 and k_2, which are gauche to one another and meet the ray SS' in the points O_1 and O_2, respectively, every line of the congruence meeting both these singular lines, and conversely, every line which meets both k_1 and k_2 being a line of the congruence. The singular lines or axes k_1 and k_2 are in general distinct, but may coincide or become imaginary. We shall first consider the case in which the axes are real and distinct.

40. In S and S' select at will a pair of corresponding sheaves of the second order Φ_5 and Φ_5', and denote the corresponding cones enveloped by these sheaves by K_5 and K_5'. These sheaves of planes generate a ruled surface of the fourth order, F_5^4, whose generators are lines of the congruence.

Through each point of k_1 and k_2 in general pass two pairs of planes of Φ_5 and Φ_5', and hence two generators of the surface F_5^4, so that these lines are double or nodal lines on the surface. The two generators through any point of either nodal line are as usual real and distinct, coincident or imaginary according as the point lies without, on, or within the cones K_5 and K_5', it having been shown in the most general case, and is equally true in this, that any singular point which lies without, on, or within one of these cones has a similar position with respect to the other. The cones K_5 and K_5' in general intercept each of the singular lines in two points, thus dividing these lines each into two segments, of which the one lies actually on the surface at every point, while the other is isolated. The four points in which the cones meet k_1 and k_2 are as usual pinch points on the surface. Thus the surface F_5^4 has in general four pinch points, of which two lie on each nodal line.

If the point O_1 in which SS' meets the nodal line k_1 lies outside the cones K_5 and K_5', then also does the point O_2 in which SS' meets the line k_2, and if O_1 lie either on or inside these cones, so also does O_2. Any plane of Φ_5 which intercepts k_1 in the point O_1 contains the line SS', and consequently intercepts k_2 in O_2. Its corresponding plane in Φ_5' also passes through the line SS', so that SS' is a generator. In general, two pairs of corresponding planes in Φ_5 and Φ_5' pass through O_1 and O_2, hence SS' is in general a double generator of the surface F_5^4. This double generator is real or imaginary according as O_1 and O_2 lie outside or inside the cones K_5 and K_5'. If the pinch points on either nodal line k_1 or k_2 be imaginary, that is if the sheaves Φ_5 and Φ_5' be so chosen that the cones K_5 and K_5' do not intercept one of these lines, two pairs of corresponding planes of the sheaves must always pass through O_1 and O_2, and consequently the double generator must in this case be real.

41. The surface F_5^4 admits then the following subforms:

(1). The pinch points are all real, the double generator is real. In this case the cones K_5 and K_5' intercept both k_1 and k_2, so that on each nodal line there is one segment which lies actually on the surface and one isolated segment, the points O_1 and O_2 appearing in those segments of the nodal lines which lie actually on the surface.

(2). The pinch points are all real, the double generator is imaginary. This subform differs from the preceding one only in that the points O_1 and O_2 appear in the isolated segments of k_1 and k_2, respectively.

(3). The pinch points are all real, while the double generator becomes torsal. Here the cones K_5 and K_5' cut the nodal lines in the points O_1 and O_2, respectively, and each of them in one other point. Thus two of the pinch points lie on the double generator, the remaining two being also real.

(4). The two pinch points on one nodal line are real, on the other imaginary, the double generator being necessarily real. In this case the cones intercept one nodal line in two points which must be different from O_1 and O_2, but do not intercept the other line. The nodal line on which the two real pinch points lie has one segment which lies actually on the surface and one isolated segment, the point O_1 or O_2, as the case may be, which lies on this line always appearing in the former segment. The other nodal line lies actually on the surface throughout.

(5). All four pinch points are imaginary, the double generator being necessarily real. The sheaves Φ_5 and Φ_5' are so chosen that the cones K_5 and K_5' cut neither nodal line. Both these lines lie actually on the surface throughout, and the double generator is real.

42. If the sheaves Φ_5 and Φ_5' which generate this surface be so chosen as to include either α_1 or α_2 as self-corresponding plane, in other words, be so chosen that the cones K_5 and K_5' are tangent to either of the singular lines k_1 or k_2, and hence so that the two pinch points on this line coincide, then will the surface F_5^4 degenerate into this self-corresponding plane and a ruled surface of the third order. For suppose, for example, that the sheaves include the plane α_1; then any line in this plane which passes through O_2 may be considered a generator of the surface, and hence the plane belongs wholly to the surface. An arbitrary straight line will always meet one generator of the surface in this plane, and can therefore meet at most three other generators without meeting every generator of the surface.

Under these circumstances the line k_1 is cut by the planes of the sheaves Φ_5 and Φ_5' in two coincident ranges of points which are perspective to these sheaves. Through each point of k_1 passes one and only one generator of the surface of the third order, since through each point of this line there passes one and but one pair of corresponding planes of the sheaves, aside from the self-corresponding plane α_1, while in general through each point of k_2 will pass two generators of this surface.

An arbitrary plane ρ through one of the generators which intersect the line k_2 in any point R cuts the sheaves Φ_5 and Φ_5' in projective sheaves of lines of the second order which have two self-corresponding rays, namely, the generator lying in the plane and that line in which the plane ρ cuts α_1. Hence these sheaves generate a conic r^2 which is met by every generator of the surface, and which consequently passes through the point R. This conic is perspective to both sheaves Φ_5 and Φ_5', and is therefore projectively related to the line k_1. Hence the surface of the third order arising from the sheaves Φ_5 and Φ_5' may be considered as having been generated by the conic r^2 and the straight line k_1, which are related projectively to each other, and lie in different planes, the line k_2 meeting the conic and appearing as a double director on the surface.

On the other hand, if the sheaves Φ_5 and Φ_5' be so chosen that both α_1 and α_2 are included as self-corresponding planes, then the cones K_5 and K_5' are tangent to both k_1 and k_2, so that the pinch points on each line coincide, and the surface generated by these sheaves breaks up into these two self-corresponding planes and a ruled quadric. The lines k_1 and k_2 are in this case both perspective to the sheaves Φ_5 and Φ_5', and are therefore projectively related to each other. These generate the ruled quadric, which is such that the line SS' cannot be a generator, since the point O_1 of k_1 corresponds to the point of contact of the cones K_5 and K_5' with k_2, and similarly, the point O_2 of k_2 corresponds to the point of contact of these cones with k_1. Moreover, it is evident that the surface F_5^4 can have no straight line director other than the two nodal lines without degenerating.

43. An arbitrary plane through a generator PQ of the non-degenerate surface F_5^4 cuts the surface along this generator and in a curve of the third order which has a double point at the intersection of the plane with the double generator. This cubic curve meets the generator lying in its plane in three real points, namely, in the points of intersection, P and Q, of the plane and generator with the nodal lines k_1 and k_2, respectively, and in a third point at which the plane cuts the consecutive generators and is tangent to the surface. If this point of tangency move along the generator till it coincide with the point P, the plane at the same time rotates about the generator till it contains the nodal line k_1. In this position the plane also contains the second generator through Q, and is tangent to the surface at the points of intersection of these two generators with

the nodal line k_1. The curve of section of this plane consists of the two generators lying in it and the nodal line k_1 counting doubly, since in each point of this line two generators are intercepted by the plane.

44. Every plane through a nodal line is a bitangent plane to the surface, since it contains two generators which intersect in a point of the other nodal line, and conversely, the plane of the two generators which meet in any point of one nodal line contains the other nodal line, and is tangent to the surface at the two points along this second line in which the two generators intersect it. The two points of tangency of the plane determined by either nodal line and the two consecutive generators which intersect in a pinch point on the other nodal line coincide, while the plane is tangent to the surface all along these generators except at the pinch point. At this point the tangent plane is determined by this torsal generator and the nodal line on which the pinch point lies.

On the other hand, at every point of either nodal line two tangent planes exist which are in general distinct, namely, the planes determined by the nodal line and each of the two generators passing through the point. Only at the pinch points and at the singular points O_1 and O_2 do these two tangent planes coincide. Although the two generators through the singular points O_1 and O_2 of this surface coincide, these points still differ from ordinary pinch points, since the two generators through them are not in general consecutive generators.

Since in general two pairs of corresponding planes of Φ_5 and Φ_5' intersect in the line SS', an arbitrary straight line which meets SS' can meet at most two other generators of the surface, hence an arbitrary plane γ through SS' cuts the surface F_5^4 in this doubly counting generator and in a conic g^2 which meets this generator in two points, and which is perspective to both sheaves Φ_5 and Φ_5'. The plane of the conic is tangent to the surface at the two points in which the conic and double generator SS' intersect, for at each of these points the plane intercepts generators which are consecutive to the double generator. Only when the arbitrary plane coincides with either of the singular planes α_1 or α_2 does the conic degenerate, in which case the curve of section consists of the double generator and the nodal line which lies in the plane, while the two points of tangency coincide at the intersection of these lines.

Thus the bitangent planes of the surface F_5^4 form three sheaves of the first order, whose axes are the nodal lines k_1 and k_2 and the double generator SS'.

45. Since each generator of the surface F_5^4 meets both straight lines k_1 and k_2 and the conic g^2, this surface may be generated by a straight line which moves so as always to meet two straight lines not lying in the same plane and a conic which neither meets nor lies in the same plane with either line. The double generator appears as the line joining the points in which the plane of the conic cuts the two arbitrary lines, and is real or imaginary according as the conic cuts this line in real or imaginary points.

46. Two arbitrary planes through the line SS' cut the surface F_5^4 in conics which are both perspective to the sheaves Φ_5 and Φ_5', and which are consequently projectively related to each other, and in such a manner that the points in which the one conic intersects the double generator correspond respectively to the points in which the other conic intercepts this line. Hence the surface F_5^4 may be generated by two projectively related conics which lie in different planes and have no point in common, but which are so situated that the points in which the one conic cuts the line of intersection of their planes correspond respectively to the points in which the other conic cuts this line. But this method of generating the surface is the reciprocal of that by which the surface was originally generated, the two projective conics determining the collinearity of the planes in which they lie in such a manner that their line of intersection is a self-corresponding line, but not every point of it a self-corresponding point. Hence the surface F_5^4 also is its own reciprocal.

§7.—*The Surface with One Double Straight Line Director and a Double Generator.*

47. Suppose now that the two self-corresponding planes in the bundles S and S' approach indefinitely near to each other and finally coincide in a single plane α_{12}. Then, as we have seen, the two singular lines k_1 and k_2 also coincide and lie in the plane α_{12} as one single line k_{12} containing all the singular points of the congruence outside the ray SS'. Through each point of k_{12} there passes an infinite number of lines of the congruence which lie in a plane with k_{12}.

Under these conditions a surface F_6^4 generated by a pair of corresponding sheaves of planes of the second order, Φ_6 and Φ_6', lying in the bundles S and S' is as before of the fourth order, and has the line k_{12} as a nodal line, while in general the line SS' joining the centres of the sheaves lies on the surface as a double generator. The cones K_6 and K_6' which are enveloped by the sheaves Φ_6 and Φ_6', respectively, intercept the line k_{12} in general in two points, both cones passing through the same points of the line. These two points are the pinch points on the surface F_6^4, and may be either real or imaginary. The pinch points as usual divide the nodal line into two segments, of which the one lies actually on the surface at every point while the other is isolated. The double generator SS' is real or imaginary according as the point O_{12} in which SS' meets the nodal line k_{12} lies actually or ideally upon the surface.

If one of the two pinch points coincides with O, then is SS' a doubly counting torsal generator. When the two pinch points on the surface are imaginary, that is, when the cones K_6 and K_6' do not intercept the line k_{12} in real points, then must the line SS' be a real double generator. If, finally, the two pinch points coincide, which happens when the cones K_6 and K_6' are tangent to the line k_{12}, then is the plane α_{12} a self-corresponding plane of the generating sheaves, and hence the surface breaks up into this plane and a ruled surface of the third order.

48. Every plane through the nodal line k_{12} contains two generators which intersect in a point of this nodal line; that is, every plane through the nodal line is a bitangent plane to the surface, the two points of tangency coinciding in a point of the nodal line. Conversely, since the two generators which pass through any point of the nodal line lie in one plane through this line, the two planes which are tangent to the surface at this point coincide. Every point along the nodal line k_{12} is therefore a cuspidal point of the surface, but it is only at the pinch points and at the singular point O_{12} that the two generators through the point coincide. Here again the singular point O_{12} differs from an ordinary pinch point, as the generators through it are not in general consecutive generators. The curve of section of any plane through the nodal line consists of the two generators lying in that plane and of the nodal line counting doubly.

49. As in the surface F_5^4, any plane through the double generator cuts the surface along this generator and in a conic perspective to both the generating sheaves Φ_6 and Φ_6', and which meets the double generator in two points at which the plane is tangent to the surface. The bitangent planes of the surface F_6^4 thus form two ordinary sheaves of the first order whose axes are the nodal line k_{12} and the double generator SS'. It readily follows, as in the surface last considered, that the surface F_6^4 is likewise of the same form as its reciprocal.

50. Of the remaining four species of the Ruled Surface of the Fourth Order enumerated by Cremona, none can be generated by means of projective sheaves of planes of the second order, or by the reciprocal method of projective conics. Two of these, Cremona's ninth and tenth, Cayley's third and sixth, are self-reciprocal, triple-line surfaces, the ninth having a distinct single straight line director, the tenth without such a director. Both these surfaces admit trinodal quartic sections, the nodes, however, being coincident, forming a triple point. The former surface may be generated by means of a straight line and a plane cubic with a double point which are projectively related to each other, on condition that the point of the straight line which lies in the plane of the cubic corresponds to that third point of the cubic which lies in the plane determined by the generating line and the double point of the cubic; the latter surface, on condition that the generating line pass through the double point of the cubic.

The remaining two surfaces, Cremona's eleventh and twelfth, Cayley's first and fourth, do not admit a trinodal quartic section. They differ from the surfaces denoted by F_5^4 and F_6^4, respectively, in that they have no double generator, but have in general four pinch points on each nodal line.

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